4. MATHEMATICAL MODELS

4.1 Models. A system may be studied by means of observation and experimentation made on the system itself. In many cases this is not possible.

Examples:

-The system may not exist (an airport to be built, the Roman Empire).

-It may be inaccessibly (the interior of a star, or the earth).

-To experiment with it is impossible (continental plates).

-Experiments are forbidden (the system have human beings or ecological elements).

-Experiments are very costly (a national economy).

-The time scale is very large or small (collision between galaxies, nuclear fission).

-The data are not available now (greenhouse effect).

In many of these situations it is possible to build and experiment with an **artificial system** similar to the real system in structure and behavior. This is called a **model** of the system.

Maps, pictures, mock-ups, pilot plants, guinea pigs, plots for experimental sowing, graphs, matrices, verbal descriptions, mental representations, fables, metaphors, systems of differential, algebraic or logical equations, has been used as models. In all these models the real system, difficult to grasp or handle, is substituted by an artificial system, made of more familiar or handleable objects and relations that constitute the model.

The similitude may be a partial correspondence between the **elements**, **relationships**, and **behavior** of the system and those of the model..

Model building is an essential process for understanding. **Models are built to know** and to solve problems with this knowledge. It is assumed that, when building a model, substantial understanding about the system is gained.

To solve a problem, requires knowing what may be the changes in the system outputs or structure (system behavior) when some changes are made in the system structure or in their inputs. The idea is to make the corresponding changes and see their effects on the model. From this, the changes in the system may be deduced. For example, a point in a map (model) indicates the real place in which I stand and I want to reach another place (problem). A road in the map indicates in the model a convenient way to reach the representation of the target point. So I select the real road corresponding to the represented road and follow it.

The main problem when using models is **the similitude problem. The behavior of the model may differ in some aspects from the behavior of the system**. So the decisions based on the model behavior may be wrong. There are methods to estimate and reduce this type of errors, i.e. to judge and increase the **validity** of the model. (See chapter 9)

The second is the **modeling problem**. It may be difficult to translate the collected information about the system into a model. Learning some patterns corresponding to different types of models may help to develop the ability to design models. In this chapter some types of **mathematical models are discussed**. The elements and realtions are represented by algebraic, integral or differential equations. The conclusion may be found by solving these equations. In some important cases this is not possible. In many of these cases the conclusions may be extracted by a **simulation process.** Simulation methods are dealt with in chapters 6,7, 8.

The third question is about use of he model or **experimentation**, which is discussed in Chapter 9.

It is clear that, for the same problem, the associated system that is considered depends to a certain extend on the people that proposes the problem and the people that analyze and try to solve it. That is to say, from the people who participates in the design of the model. Different analysis and different models will result when different people, or the same people in different situations, perform them. This relativity of the models has been pointed out by Kuhn in the case of scientific models [Kuhn, 1962]. It is important some consideration of the professional, organizational and social environment in which the model is designed, to be aware of possible biases, omissions, and inclusions of elements and relations in the model.

4.2 Dynamic mathematical models. This text deals with dynamic mathematical models. In them the properties and relations of the real system and the inputs and outputs are represented by **mathematical variables, constants and functions**. The **values** of the variables and constants may be **arithmetic**, **logical**, **statistical distributions**, **interval valued**, or **elements** of an arbitrary predefined set.

In **dynamic** model the values of the variables change with the values of **time**, a real special variable that take increasing values. In general the topological and algebraic properties of the variables depend on the type of scale (see 3.1). Each variable may be simple (one valued) or a vector, matrix or tensor or, in general, an array. Algebraic, differential or integral equations, tables or algorithms may express the functions. Those functions are used to give values to certain variables when the values of some others variables are known or to find future values of some variables when the actual values are known. Random variables may also be used, an in these cases the values are specified by **probability distributions** and they may be assigned as the result of a stochastic (or pseudo-stochastic) process (see Chapter 6).

The dynamic character of the models means that the values of the variables, and in some cases their relationships change with time. In the case of random variables **the distribution functions** may change with time.

Four types of models have been developed to facilitate the building of a mathematical dynamic model from an informal description. Each type deals with the **time** in a different way and it differs in the **method to draw conclusion**s from the model, i.e. the way in which the values of the variables are computed. They are:

Solution Methods

Analytical

Algorithmic

Time	Continuous	**Differential Equations models**	**Continuous models** Values of variables computed at equal time intervals
	Discrete	**Difference Equations models**	**Discrete event models** Values of variables are computed at events in which there are changes

1. **Differential equations**, in which the time changes **continuously**, and the values of the variables i.e the functions of time, are determined **analytically** by solving the differential equations. General solutions are obtained and all the particular solutions can be obtained changing the parameters in the general solution. From the analysis of the general solution may be extracted many properties of the behavior of the model.

2. **Difference equations,** in which the time changes by **steps** of equal size, and the values of the variables are computing **analytically** by solving the difference equations. General solutions are obtained and all the particular solutions can be obtained changing the parameters in the general solution. From the analysis of the general solution may be extracted many properties of the behavior of the model.

3. **Continuous simulation**, in which the successive values of the variables are computed by **algorithms**, at **equal** time intervals. In these models, like in the finite difference models, the time is considered to change in equal jumps of duration h. The value of the time is computed by $t_{i+1} = t_i + h$. The values of **all the variables** at the instant t_{i+1} are computed from the values of the state variables and the exogenous variables at the instant t_i. That is to say: using the values of the variables at the beginning of one interval and the equations of the model, the values at the end of the interval are computed. These values are taken as the values of the beginning of the following interval and the computing process is repeated. **The output** of the model is not an analytical solution formula that gives the values of the variables as a function of the time (as in finite difference models) but a **time series** for each output variable. In chapter 7 they will be discussed in

detail. This type of simulation is usually called **continuous simulation**. See Roberts and others 1983.

The name “continuous”, usual in the literature, is not appropriated since the computing is made only for a discrete set of values of time. The method is rather an algorithmic solution of a set of difference equations. The solution obtained in one computation (run of the model) is a time series of values and corresponds to the particular values of the parameters for that run. To obtain solutions for different values of the parameters another run of the model may be done.

4. **Discrete event simulation**, in which the successive changes of the variables are computed by **algorithms** at **unequal** time intervals. It is assumed that the variables change only at certain instants called **events.** The difference with the previous type is that the interval between successive events are not equal. In these events the values of **some variables** are changed and some events are scheduled for future times. Only the variables are considered that change at each particular event and the intervals between the successive events are not necessarily the same. The solution obtained in one computation (run of the model) is a time series of values and corresponds to the particular values of the parameters for that run. To obtain solutions for different values of the parameters another run of the model may be done. . These models are discussed in detail in chapter 8. See Law & Kenton 1991).

It is possible to combine continuous and discrete event simulation in the same computing schema. The difference equations may be computed in equidistant events that are mixed with the unequally distant discrete events. Some simulation languages use this technique (see Domingo et al. 1992)

In this chapter the first and second types are briefly considered, whereas the other two types are considered in detail in the rest of the book. In this chapter and the next one some important dynamic and non-dynamic models are discussed because they can be used as components of complex dynamic simulation models. In the discussion of the models the important system concepts of **state, phase space, stability of equilibrium, oscillations, resonance, control, delays, catastrophes, chaos, and entropy**, are introduced in an intuitive way.

4.3 DIFFERENTIAL EQUATION MODELS.

4.3.1 Expression of models by differential equations

In these models the speed of change of some variables are expressed as a function of other variables (and perhaps also the time). Some variables describe properties of the elements of the system or relations among them (**endogenous variables**), other describe the inputs (**exogenous variables**) and other the outputs. In these models it is possible to identify a minimum set of endogenous variables whose values at a time can determine the future values of these and all the other endogenous variables of the model. These special variables are called **state variables**. The set of values of the state variables at a given time is called **state** of the system at that time.

If the **state variables** are designed by the time functions:

$x_1(t), x_2(t), \ldots, x_n(t)$,

the exogenous variables as:

$e_1(t), e_2(t), \ldots, e_m(t)$,

and the other variables (**auxiliary, non state variables, and some outputs**) as:

$y_1(t), y_2(t), \ldots, y_k(t)$

then the model is:

$$\frac{dx_i}{dt} = f_i(\ x_1(t), x_2(t), \ldots, x_n(t)\ , e_1(t), e_2(t), \ldots, e_m\ (t)\ ) \qquad i = 1,2,\ldots,n \quad (1)$$

$$y_j = g_j(\ x_1(t), x_2(t), \ldots, x_n(t)\ , e_1(t), e_2(t), \ldots, e_m\ (t)\ ) \qquad j = 1,2,\ldots,k \quad (2)$$

where: $\frac{dx_i}{dt} = x_i'(t)$ is the speed of change of x_i .

In the functions f_i and g_j not all the state or exogenous variables must necessarily appear.

The exogenous variables $e_l(t)$ must be given as explicit functions of time by means of algebraic expressions, graphs or tables with interpolation rules. The functions f_i and g_j are given and they describe the structure of the model. The functions $x_i(t)$ must be determined by solving the system (1) of differential equations. Then the $y_j(t)$ can be calculated by (2).

The differential equations may be solved analytically by methods indicated in standard books on analysis or differential equations. Some mathematical packets as MAPLE or MATHEMATICA allow the solution of many types of differential equations. If there is not analytical solution (as happen quite often) the differential equations may be numerically solved. Many books deal with the numerical methods (Forsythe, Malcom, Moller 1977, Stoer & Bulisrt 1980) .

If the model contains some differential equations of higher order, these may be substituted by a number of first order differential equations, the number being equal to the degree of the original equation.

Example: Let it be the third order differential equation:

$$y''' - 3y'' - y' + 9y = 0$$

Introducing the variables: $g = y''$ $h = y'$ the equivalent system of three first order differential equations is:

$$y' = h$$

$$h' = g$$

$$g' = 3g + h - 9y$$

4.3.2 Examples (See Dreyer 1993; Burghes & Borrie 1981)

a) Exponential growth and decrease.

It often happens in physical, biological, and economic processes that **the growth or decrease of a quantity may be proportional to that quantity**. The rate of increase of a population may be proportional to that population (the annual increase in a city of 50000 inhabitant may be twice the increase in a similar city of 25000 inhabitants). The annual increase of a capital at a constant rate of interest is proportional to the capital. The rate of decrease of the mass of a radioactive substance that disintegrates is proportional to its mass. The differential equation for the growth of the quantity y is:

$$y' = \frac{dy}{dt} = ky \qquad (3)$$

This equation that express the whole model is solved by the standard procedure of separation of variables:

$$\frac{dy}{y} = k\,dt$$

by integrating both sides results: $\int \frac{dy}{y} = k \int dt$ $\quad \ln y = kt + c$ and:

$y = e^{kt+c}$

$y = y_0 e^{kt}$

where: $y_0 = e^{kc}$ represents the initial value of y when $t = 0$. See Figure 1.

The constant $k = \frac{dy}{dt} / y$ represents the increase (decrease) of y per unit time and per unit of y.

Example 4-1 If the equation is $y' = 0.015040774 \times y$ and for t=0 is y=4 we have the solution:

$y = 4e^{0.01504774\,t}$. See Figure 4-1

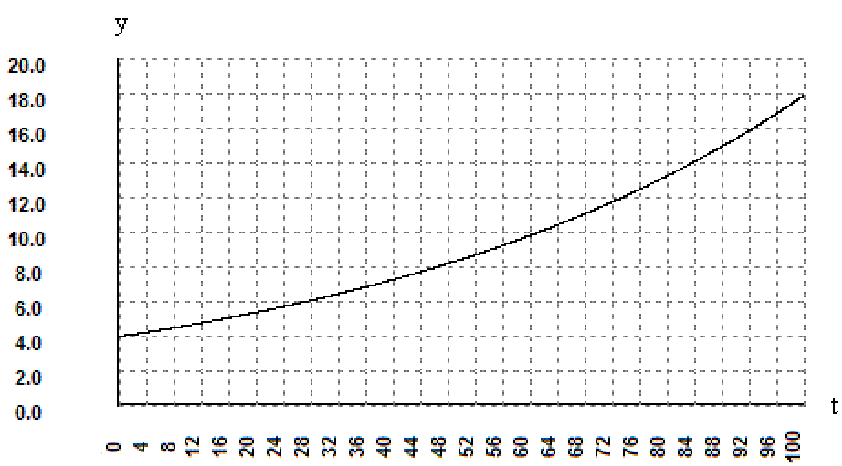


Figure 4-1 Exponential growth

Example 4-2 If the population p of a country is 23 millions and the growth rate is $k = 0.026$ (for each person there is an increase of 0.026 persons each year (or for each 1000 persons there is an increase of 26 persons each year)) then , the law of growth is $p = 23000000e^{0.026\,t}$, where t is the time in years counted from the initial year. Or if the initial year is the beginning of 1996 then:

$p = 23000000e^{0.026\,(t-1996)}$

where: t is now a year equal or greater than 1996 and p the population at the beginning of such year.

In the case of a decrease proportional to the quantity the model is:

$$\frac{dy}{dt} = -ky \qquad (4)$$

and the law of decrement is the solution of (4)

$y = y_0 e^{-kt}$

in which y_0 is the initial quantity and $y \to 0$ as $t \to \infty$

Example 4-3: If the equation is $y' = -0.01504966 \times y$ and for t=0 is y=20 we have the solution:

$y = 20 e^{0.01504966\, t}$. See Figure 4-2

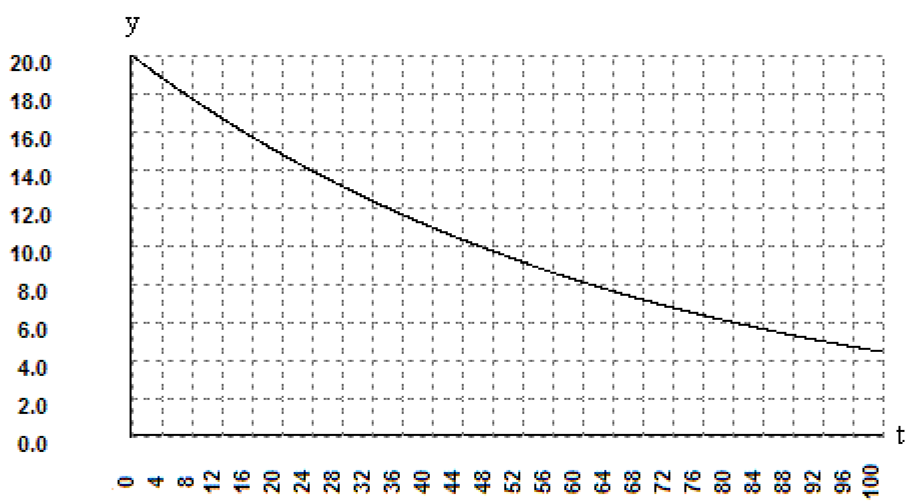


Figure 4-2 Exponential decreasing

Example 4.4:

A cobalt radioactive source used for in medicine has now a quantity of 0.2 g of the radioactive isotope Co^{60} whose disintegration constant is 0.1216g/year. How much Co^{60} will remain after 12 years.

$c = 0.2 e^{-0.1216 \times 12 = 0} = 0.0465 g$

In the exponential growth the quantity duplicates each ln2/ k units of time (see Exercise 1). In exponential decay it reduces to a half in such a time.

b) Decrement proportional to quantity plus constant supply. An usual case is an exponential decreasing compensate by a constant supply. The model is:

$$\frac{dy}{dt} = p - ky$$

where p is the rate of supply (supplied quantity each unit of time) and k the rate of decay of each unity of quantity per unit of time.

If at $t = 0$ the quantity is 0 the speed of growth at the beginning is p . As y becomes greater the speed of growth decreases as y approaches to $y = p/k$, then the speed of growth tends to 0 and y tends to be constant p/k . The quantity tends to the equilibrium value p/k .

The solution of the linear equation with the initial condition: for $t = 0 \quad y = y_0$ is:

$$y = (y_0 - \frac{k}{p})e^{-kt} + \frac{k}{p} \qquad (5)$$

Notice that when $t \to \infty \quad y \to p/k$ independently of the initial condition.

In a more general model the supply may be a function $p(t)$.

Example 4-5:

Radiocarbon data. (Libby 1948)The neutrons produced by cosmic radiation transform some nitrogen atoms of the atmosphere into radioactive carbon C^{14}: N^{14}+n=C^{14}+ p^+. The proton p^+ takes an electron forming hydrogen, while the C^{14} combines with oxygen forming a CO_2 molecule with the carbon atom radioactive. This C^{14} disintegrates with a constant equal to 0.00012097. However, as it is constantly produced, the proportion of C^{14} in the atmosphere is constant because it must have reached the equilibrium quantity in the long run, as was shown by equation (5). The living plants, that continually absorb C from the air by photosynthesis, must have in their tissues the same proportion of C^{14}. When a plant is cut the C^{14} in their tissues decays but is not replaced by new C^{14} from the photosynthesis. So there is a process of simple exponential decrease of C^{14}. If the proportion of C^{14} in a plant cut many years ago is determined, then the time in which the plant was cut may be determined. Assume that in a counter device, 160 counts by minute from the C^{14} from normal CO_2 in the air are detected, due to the electrons emitted by the radioactivity. When this CO_2 is replaced in the counter by the CO_2 from burning a piece of an ancient furniture (taken from a well dated Egyptian tomb of about 4300 years ago) 95 counts are obtained because the radioactive C^{14} has disintegrated since the wood was cut.

The number of counts is proportional to the concentrations of C. So that: $95 = 160e^{-0.00012097\ t}$

$$t = \frac{\ln(160/95)}{0.00012097} = 4309\ .$$

The wood was extracted from the tree 4309 years ago.

c) Logistic model. Usually an exponential increase cannot last a long time (see Exercise 2). It is usual that the rate of growth k decrease with the quantity y. The simplest assumption is that k is a linear decreasing function of y, so that instead of k the rate of growth may be $k(1-y/y_m)$ where k and y_m are constants. This expression is equal to k when $y=0$, and is 0 when $y=y_m$, decreasing linearly between these values. The differential equation model is:

$$\frac{dy}{dt}=k(1-\frac{y}{y_m})\ y$$

Note that when y is small respect to y_m the equation is approximately equal to that of exponential growth, so that an almost exponential growth may be expected in this early growth. As y increases the rate of growth decreases, and when y tends to y_m the rate of growth tends to zero and the y tends to the constant value y_m.

The equation can be analytically solved by variables separation. It gives:

$y=\frac{y_m}{1+Ce^{-kt}}$ which has the value $\frac{y_m}{1+Ce}$ when $t=0$ and $y\to y_m$ when $y\to\infty$

Example 4-5 If the equation is $y'=0.1500387\times(1-\frac{y}{30})y$ and y=2 for t=0 then the solution is $y=\frac{30}{1+16e^{-0.1500387\,t}}$

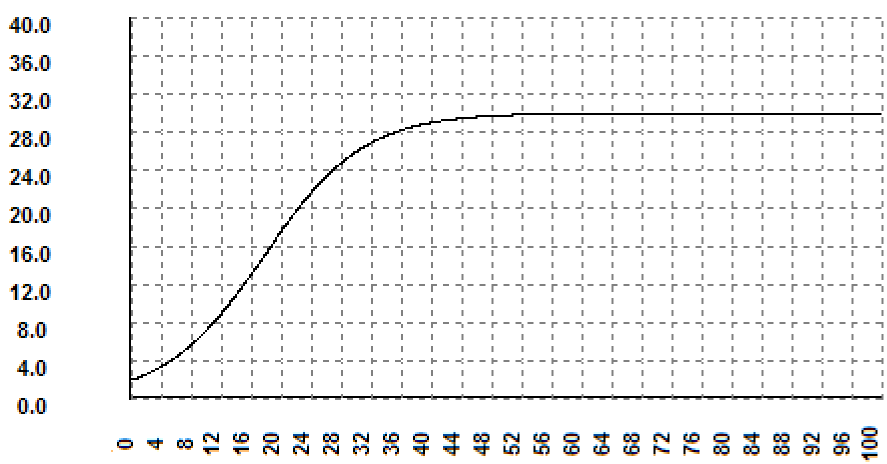


Figure 4-5 Logistic growth

This curve is called **logistic** or Verhulst curve from the biologist that introduced it. This law of growth describes many biological and economic processes fairly well. See Exercises 3.

d) Model Prey predator. **State Space**. (See Kemeny & Snell 1962, Maynard Smith 1974) One model of this type was developed by Lotka and Volterra near 1930. In this model the predators (that will be identified as foxes) feed of an only type of prey (that may be vegetarian rabbits). The quantities of foxes and rabbits will be denoted by f and r and will be considered continuous real variables. The growth law of the rabbits (without foxes) may be expressed by $\frac{dr}{dt} = c_1 r$ that corresponds to an exponential growth. Because the actions of the predator will be a rate of elimination of rabbits that may be assumed proportional to the product $r\,f$ of the quantity of rabbits and foxes. This law is justified noting that the number of encounters of rabbits and foxes is proportional to the quantity of both. So the model for the rate of change of rabbits is:

$$\frac{dr}{dt} = c_1 r - c_2 rf \qquad (1)$$

If the population of rabbits has limits (because of lack of enough food) and its free growth is logistic instead of exponential the equation is:

$$\frac{dr}{dt} = c_1(1 - \frac{r}{r_m})\, r - c_2 rf \qquad (1')$$

where r_m is the maximum number of rabbits that may sustain the region without predators.

The predators, if there were not rabbits, would decrease exponentially by starvation

$$\frac{df}{dt} = -c_3 f$$

but thanks to the rabbits, this is balanced by a rate of births that is assumed to be proportional to the rate of feed rf . A growth term must be added. Thus the model for the foxes is:

$$\frac{df}{dt} = -c_3 f + c_4 rf \qquad (2)$$

The **model** consists of the system of differential equations (1) and (2) or (1') (2). The **state variables** are r and f .

State space. Before considering an explicit solution, it is important to see a graphical representation of the process. If r and f are taken as coordinate axis, one point in the plane will represents a value of r and a value of g, so, it represents a **state** of the system. A movement (trajectory) of this point will represent the evolution of the system.

The space whose coordinates are the state variables is called the **state space** (some authors call it **phase space**, but this name is used for the physicists to design the space whose coordinates are the state variables and its derivatives respect to time).

Starting from one point at time 0, the point representing the state will move in a certain trajectory. Starting from another point a different trajectory may be obtained. Two different trajectories cannot intersect each other, because in a deterministic model one state cannot be followed by different states. The differential equation of the trajectories may be obtained dividing (2) by (1):

$$\frac{df}{dr} = \frac{-c_3 f + c_4 fr}{c_1 r - c_2 fr}$$

This equation can be integrated by separating variables (see Exercise 4).

$f^{c_1} e^{-c_2 f} = k r^{-c_3} e^{c_4 r}$ where f cannot be explicitly expressed as a function of r.

To find points (r, f) an iterative process may be used. See Figure 3.

Equilibrium and stability. The system is in equilibrium when the state variables f and r do not change with time (their derivative is 0). From (1) and (2) the conditions for this case are obtained:

$$c_1 r - c_2 rf = 0$$
$$-c_3 f + c_4 rf = 0$$

An obvious solution is:

$$r = 0, \quad f = 0 \qquad (3)$$

Disregarding this case for the moment, it is possible to divide the first by r and the second by f and it results $c_1 - c_2 f = 0 \quad -c_3 + c_4 r = 0$

so that $r_e = c_3 / c_4$ and $f_e = c_1 / c_2$ (4)

are the quantity of rabbits and foxes that are in equilibrium (note that it is a dynamic equilibrium; the animals are born and death but their quantities remain constant).

It is interesting to investigate some properties of the equilibrium points. The most important property is **stability.** This concept may be explained by a mechanical example. A ball in a bowl may be in equilibrium when the ball is at the lowest point. If it is move away a little from this point and left free, it will move toward the equilibrium position. If the ball is at equilibrium on a sphere, when moved away from the equilibrium point, it will run further away from the equilibrium point. In the first case it is said that the equilibrium is **stable** in the second it is said **unstable.** In the stable case the ball may move in different ways near the equilibrium point. The ball may move directly to it, or may **rotate** around it, or approach to it describing a **spiral.**

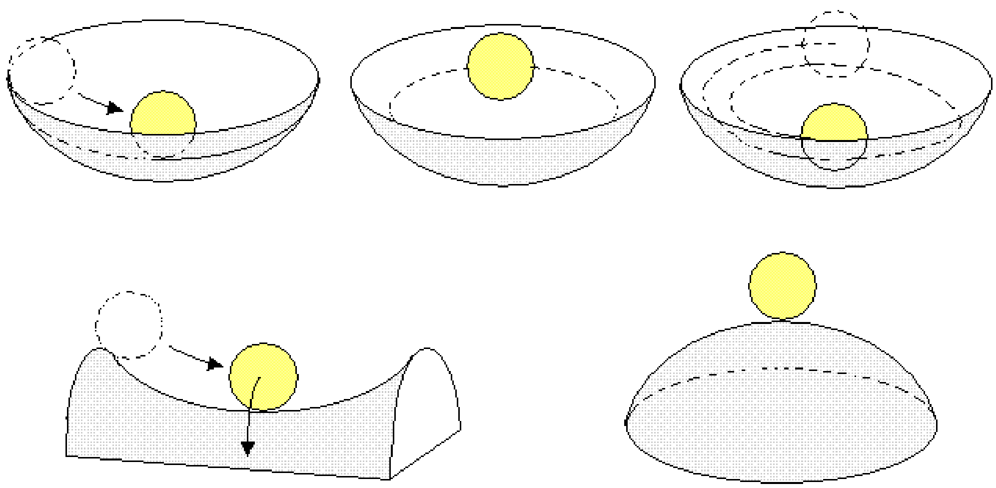


Figure 4-6 Types of equilibrium

If the ball is on the center of a **saddle** shaped surface and is moved toward the ascending part, the ball will tend to the equilibrium point, but if it is moved toward the descending part it will move away from equilibrium.

In the case of the state point at the $r\,f$ plane the equilibrium at $r = 0, f = 0$ is of the saddle class. If the point is move a little along the f axis (some foxes and no rabbits) the evolution of the system is towards the origin (the quantity of foxes tends exponentially to 0). If it is moved along the r axis (some rabbits and no foxes) the point will move away from the origin as the number of rabbits grow exponentially.

To see the behavior near the point $r_e = c_3 / c_4$, $f_e = c_1 / c_2$ a change of coordinates will be made, by taken the equilibrium point $(r_e\, f_e)$ as the origin of coordinates. The new coordinates will be:

$u = r - r_e$ and $v = f - f_e$. Substituting the values of $r = u + r_e$ and $f = v + f_e$ obtained from these equations into (1) and (2) it is obtained:

$$\frac{du}{dt} = c_1 u + c_1 r_e - c_2 (uv + uf_e + vr_e + r_e f_e)$$

and considering the values of r_e and f_e from (4):

$$\frac{du}{dt} = c_2 f_e u + c_1 c_3 / c_4 - c_2 (uv + uf_e + vr_e + c_3 c_1 / c_4 c_2)$$

neglecting the product uv near the new origin:

$$\frac{du}{dt} = -c_2 r_e v$$ and, in the same way:

$$\frac{dv}{dt} = c_3 f_e u$$ is obtained.

The equation of the trajectories in terms of u and v is found by dividing:

$$\frac{dv}{du} = -k^2 \frac{u}{v}$$ where $k^2 = f_e c_3 / r_e c_1$

by separating variables and integrating: $v^2 = -k^2 u^2 + e^2$ where e^2 is the integration constant.

$$\frac{u^2}{e^2 / k^2} + \frac{v^2}{e^2} = 1$$

This is the equation of a ellipse with semi-axis e / k and e. For each k there is a different ellipse. Thus, in the neighborhood of the equilibrium point r_e, f_e, the state point moves on an ellipse, and the values of r and f oscillate around the equilibrium values.

Should the equation for the rabbits have been (1´) the state point would move in an spiral approaching asymptotically to the equilibrium. In other dynamic models the trajectories may approach to a limit circle. See Exercise 5. See Figure 4.

e) Oscillations and resonance. Second order linear differential equations.

In some systems, specially in Mechanics, the laws come in terms of "rate of change of rate of change" so that second order differential equations are the natural expression for these systems.

Although the can be transformed in a set of two first order differential equations, in many cases the direct solution lead to many interesting concepts.

Example:

A system consists in a mass of magnitude m hanging from a spring of constant k (a force of magnitude k produce an enlargement 1). See Figure 6. A small displacement y from the equilibrium position produce a force ky toward that position. The friction force is proportional to the speed: $s\,\dfrac{dy}{dt}$. From Newton´s law: mass*aceleration= total force:

$$m\frac{d^2y}{dt^2} = -s\frac{dy}{dt} - ky \quad \text{or:}$$

$$m\frac{d^2y}{dt^2} + s\frac{dy}{dt} + ky = 0 \qquad (5)$$

This is a linear differential equation of the second order (it has a second derivative)

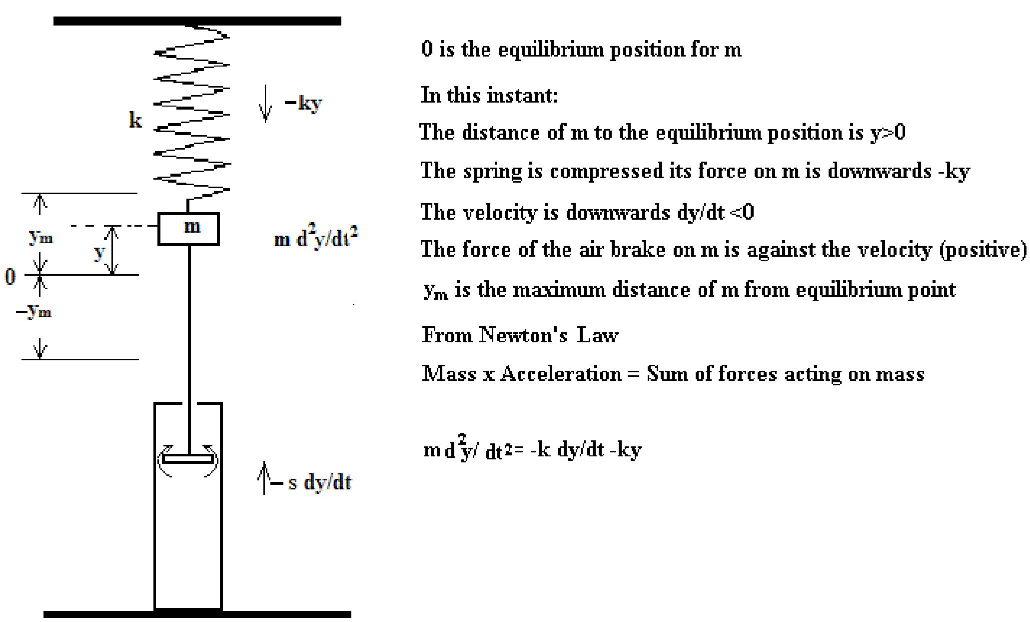
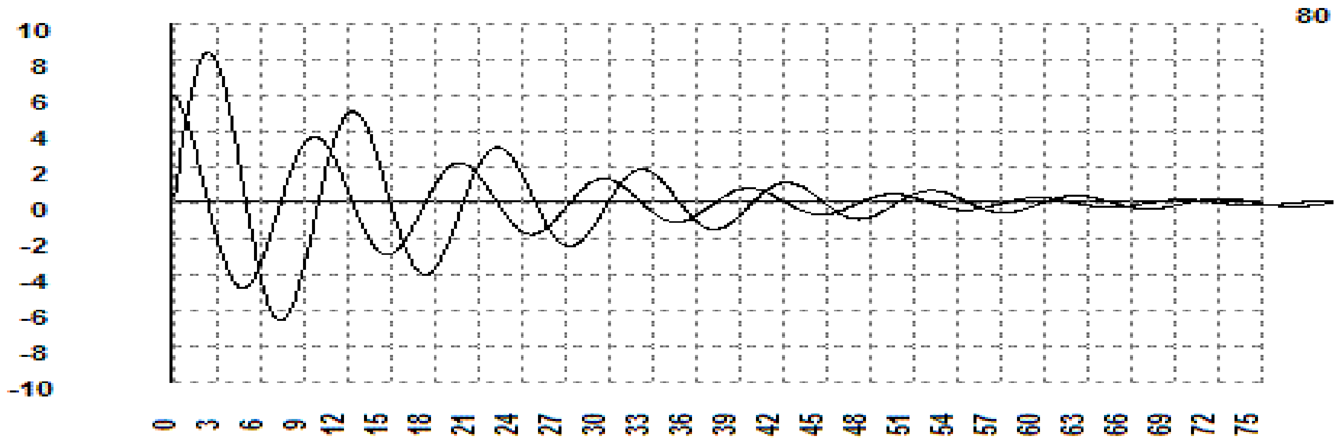


Fig. Oscilation system with air brake



Initial position y=6 Initial speed dy/dt =0 m=1 k=0.4 s=0.1

A solution of this equation is of the form: $y = e^{rt}$. Substituting in (5) and dividing by me^{rt} the condition on r is obtained:

$$r^2 + \frac{s}{m} r + \frac{k}{m} = 0$$

The values of r are: $r = -\frac{s}{2m} \pm \sqrt{\frac{s^2}{4m^2} - \frac{k}{m}}$

There are two values of r denoted r_1 and r_2 that satisfy the conditions according to the chosen sign $(+ \text{or} -)$.

$$r_1 = -\frac{s}{2m} + \sqrt{\frac{s^2}{4m^2} - \frac{k}{m}} \quad \text{and} \quad r_2 = -\frac{s}{2m} - \sqrt{\frac{s^2}{4m^2} - \frac{k}{m}}$$

It is easy to see (Exercise 6) that a linear combination of solutions is a solution. So, the most general solution is:

$$y = Ae^{(-\frac{s}{2m} + \sqrt{\frac{s^2}{4m^2} - \frac{k}{m}})t} + Be^{(-\frac{s}{2m} - \sqrt{\frac{s^2}{4m^2} - \frac{k}{m}})t} = Ae^{r_1 t} + Be^{r_2 t}$$

where A and B are constants that can be calculated from the initial conditions. If this conditions are: $y = y_0, \frac{dy}{dt} = 0$ for $t = 0$ (the mass is move away a distance y_0 from the equilibrium an them released there without any speed), then :

$A + B = y_0$ and since $\frac{dy}{dt} = r_1 A e^{r_1 t} + r_2 B e^{r_2 t}$; $r_1 A + r_2 B = 0$; from this:

$$A = \frac{y_0 r_2}{r_2 - r_1} \quad B = -\frac{y_0 r_1}{r_2 - r_1}.$$

The type of solution depends on the value of $D = \frac{s^2}{4m^2} - \frac{k}{m}$.

If $D > 0$ (because s is large) then both $r_1, r_2 > 0$ and both terms of the solution decrease exponentially, which corresponds to a damped movement.

If $D = 0$ then $r_1 = r_2 = r$ and the two parts of the solution are not independent, but it can be shown that te^{rt} is a solution (see Exercise 7) So, the general solution is:

$y = Ate^{-rt} + Be^{-rt}$ that increases for small values of t and then decreases almost exponentially as the increase of t is offset by the rapid decreasing of e^{-rt}. The values A and B computed above do not apply to this case (see Exercise 9).

If $D < 0$ then both r_1, r_2 are complex and can be denoted by:

$r_1 = a + ib$ and $r_2 = a - ib$, where: $a = s/2m$ and $b = \sqrt{\frac{k}{m} - \frac{s^2}{4m^2}}$.

and the general solution is:

$$y = Ae^{(a+ib)t} + Be^{(a-ib)t}$$

recalling Euler's formulae $e^{ibt} = \cos bt + i \operatorname{sen} bt \qquad e^{-ibt} = \cos bt - i \operatorname{sen} bt$

$$y = Ae^{at}(\cos bt + i \operatorname{sen} bt) + Be^{at}(\cos bt - i \operatorname{sen} bt)$$

to eliminate the imaginary part define $A = P - iQ$ and $B = P + iQ$

and the expression becomes:

$y = e^{-at}(C_1 \cos bt + C_2 \operatorname{sen} bt)$ where $C_1 = 2P$ and $C_2 = 2Q$

$$y = e^{-\frac{s}{2m}t}\left(C_1 \cos\sqrt{\frac{k}{m} - \frac{s^2}{4m^2}}t + C_2 \operatorname{sen}\sqrt{\frac{k}{m} - \frac{s^2}{4m^2}}t\right)$$

which represents a damped oscillation with period : $T = \frac{2\pi}{\sqrt{\frac{k}{m} - \frac{s^2}{4m^2}}}$

The constants C_1 and C_2 are determined from the initial conditions (see Exercise 8).

For undamped oscillations is $s = 0$ and the period is: $T = \frac{2\pi}{\sqrt{\frac{k}{m}}}$.

Forced oscillations. Resonance.

When a time dependent force $F(t)$ acts on the mass, it must be added to the second member of (5) and:

$$m\frac{d^2y}{dt^2} + s\frac{dy}{dt} + ky = F(t) \qquad (6)$$

A general solution of this inhomogeneous equation can be found adding to a particular solution y_P a solution y_H of the homogeneous equation (5) (see Exercise 9).

Assuming an oscillating force: $F(t) = F_0 \cos \omega t$ (ω is the number of oscillations per unit time $\omega = 2\pi / T$) it can be seen by direct substitution that a particular solution is $y_P = C\cos(\omega\ t - d)$ where C and d are constants that can be expressed in terms of the parameters of the problem in the following way:

$y_P' = -C\omega\ \mathrm{sen}(\omega\, t - d)$ and $y_P'' = -C\omega^2\ \cos(\omega\, t - d)$

substituting in the (6), developing the sums of sinus and cosines and putting the resulting coefficients of sinus and cosines equal to zero it is found:

$$C = \frac{F_0}{m\sqrt{(\omega^2 - \frac{k}{m})^2 + \frac{s^2}{m^2}\omega^2}} \quad \text{and} \quad d = \mathrm{arctg}\frac{\frac{r}{m}\omega}{\omega^2 - \frac{k}{m}}$$

so, the general solution is $y = y_P + y_H$:

$$y = \frac{F_0}{m\sqrt{(\omega^2 - \frac{k}{m})^2 + \frac{s^2}{m^2}\omega^2}} \cos(\omega\, t + d) + e^{-\frac{s}{2m}t}(C_1 \cos\sqrt{\frac{k}{m} - \frac{s^2}{4m^2}}t + C_2 \,\text{sen}\sqrt{\frac{k}{m} - \frac{s^2}{4m^2}}t)$$

the second term decays with increasing t. The first represents an oscillation of the same frequency of oscillation ω than the external force but de-phased a fraction d of a period.

The maximum amplitude of the oscillation happens when $\omega^2 = \frac{k}{m}$ that is, when the period is:

$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{k/m}}$ equal to the proper period of the oscillating system in the case of undamped oscillations. In this case, when s is small respect to m, the amplitude of the forced oscillation may be very large. This phenomenon is called **resonance,** and produces remarkable effects in oscillating systems: when a not much damped oscillatory system (small s) is forced to oscillate by a periodic force with a period near the proper period of the system the amplitude may become very large, sometimes with catastrophic results. The effect comes from applying pushes of external force that reinforce each T, the proper oscillation time of the system. Seismic waves make a building oscillate. If the period of the waves is near the period of the building resonance occurs and the damage may be great. On the other hand, the resonance is essential in the electric oscillating circuit (see Exercise 10 and 11) in which external electromagnetic waves induce currents that are only large when the incoming wave has a period near the resonance period of the circuit. This allows a radio to select one from the multitude of incoming waves. See Figure 9.